

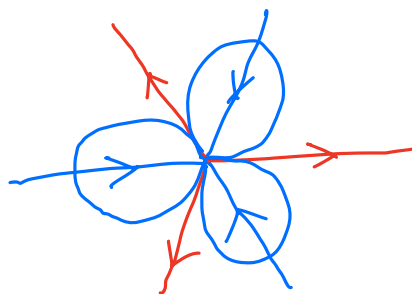
Parabolic Fixed Point

$f(z_0) = z_0, \lambda = e^{2\pi i \frac{p}{q}}, f^q(z) = z(1 + \underline{a} z^{\underline{u}} + \dots),$

$\theta = \arg(a), \underbrace{u \cdot \theta_i^+}_{\text{Attracting}} = -\theta, \underbrace{u \cdot \theta_i^-}_{\text{Repelling}} = \theta$

Thm. A) $u = u \cdot \frac{q}{p}$

B)



1) $P_i^+ \subset F(f)$

2) Small point at z_0 , tangent to repelling

3) Rotated by P/q

4) Absorbs trajectories tangent to θ_i^+

5) Every traj $\rightarrow z_0$ enters a petal. (follow from 1-4) implies

c) Repelling petal

d) P attractio,

$\alpha: P \rightarrow \alpha(P) \subset \mathbb{C}$

$P \xrightarrow{f} P$

$\alpha \downarrow \quad \downarrow \alpha$

$\mathbb{C} \rightarrow \mathbb{C}$

$w \mapsto w+1$

write out before

I. $\lambda = 1, u = 1$; II. $\lambda = 1, u > 1$; III. General λ

Proof. I: $f(z) = z(1 + az + \dots)$

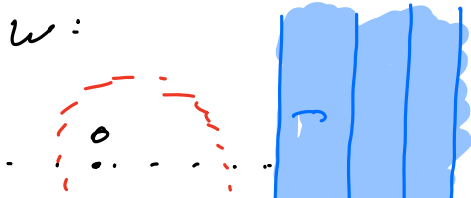
1.1) Conjugate f to make $a=1$. Will rotate the set, repelling direction

1.2) $w = \phi(z) = -1/z$

$F(w) = (f \circ \phi \circ \phi^{-1})(w) = w + 1 + O(\frac{1}{|w|})$

$H_+ = \{ \operatorname{Re} z > + \}$ forward invariant $f + \text{large}$

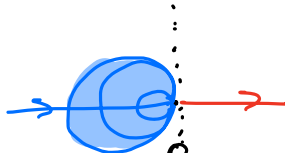
$w:$



$\phi^{-1} \quad z:$

$\rightarrow$

$\leftarrow$



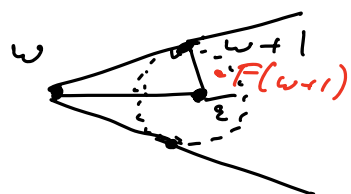
near P tangent to

$\text{Re } z = t$

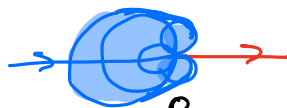
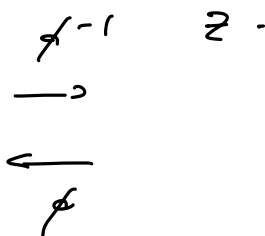
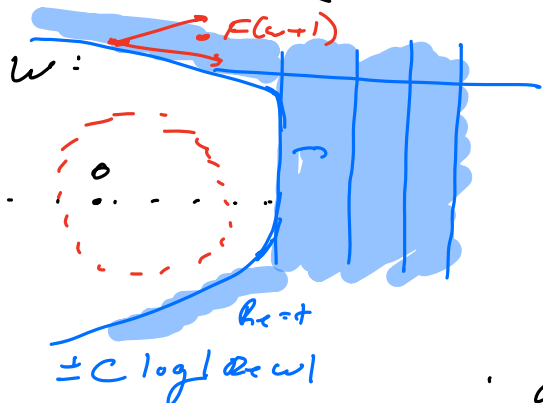
Protein of
Coulomb

resp. and together A be horizontal.

$$I.3) |F(w) - (w+1)| \lesssim \frac{1}{|w|}$$



$F(w+1)$ lies in a wedge that is narrower than π , slope ≈ 2 .



Slope of wedge is $<$ slope of log curve, so the region is convex

Log is asymptotically horizontal, so any two

1) ✓

2) ✓

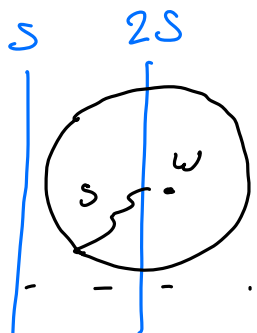
3) ✓

4) Don't see any horiz.

5) Values from the overlapping of petals ✓

$$I.4) D) \phi_k(w) = F^k(w) - F^k(w_0) \text{ for some } w_0,$$

$$\text{Re } w > 2S. |F(w+1) - (w+1)| \lesssim \frac{1}{|w|} \lesssim \frac{1}{S}$$



$$w \mapsto F(w+1) - (w+1)$$

sketch $\Rightarrow$

Skip



$$|F'(w) - 1| \lesssim \frac{1}{S^2}$$

$$w_1, w_2: \left| \frac{F(w_1) - F(w_2)}{w_1 - w_2} - 1 \right| \lesssim \frac{1}{S^2}$$

$$\left| \frac{\phi_{k+1}(w)}{\phi_k(w)} - 1 \right| \lesssim \frac{1}{k^2} \Rightarrow \left| \frac{\phi_{k+1}(w)}{\phi_k(w)} \right| \leq 1 + \frac{c}{k^2}$$

$$|\phi_k(w)| = \prod_{i=0}^k \left| \frac{\phi_{i+1}(w)}{\phi_i(w)} \right| \leq |w - w_0| \prod_{i=1}^{\infty} \left(1 + \frac{c}{i^2} \right),$$

$$\text{blue} + \text{green} \Rightarrow |\phi_n(w) - \phi_{n-1}(w)| \lesssim \frac{|w - w_0|}{n^2}, \dots$$

$\phi = \lim_{n \rightarrow \infty} \phi_n$ exists, prob. *Is onto: bijective?*

$\alpha(z) := \phi(w)$, easily verifs. D) ✓

II: $f(z) = z + az^{n+1} + \dots$

1) Conj $\leadsto f(z) = z + z^{n+1} + \dots$

2) $\zeta = z^n$, immediate in a wedge

$$f: \zeta \mapsto \zeta + n\zeta^2 + O(|\zeta|^{2+\frac{1}{n}})$$

- Remove n in conj as before.

3) $\zeta = \frac{1}{w}$

$$f: w \mapsto w + 1 + O(|z|^{-1/n}), \text{ same as I.}$$

III: $\lambda \neq 1$. Perm to f^q

Thm. Every cycle of immediate parabolic basin contains a critical point.

Proof. Iterate $f(A_0(z_0, \theta_i^+)) = A_0(z_0, \theta_i^+) = P_i^+ = P$

- If f has no crit pt, then $f: A_0 \rightarrow A_0$ is a covering map.

$$A_0 = \bigcup_{n=1}^{\infty} \underbrace{f^{-n}(P)}_{\text{conn. comp. contain } P}_0, \text{ and } P \cong D, \text{ so } f_A^{-n}(P)_0 \cong D.$$

This essentially shows if disk is simply-connected, so $f: A_0 \cong D \xrightarrow{\sim} A_0 \cong D$ is an isom.

$$A_0 \xrightarrow{f} A_0 \quad \text{since } \alpha: P \xrightarrow{\sim} \alpha(P) \supset H_f, \text{ and}$$

$\begin{array}{ccc} \downarrow \alpha & & \downarrow \alpha \\ \mathbb{C} & \xrightarrow{\quad} & \mathbb{C} \\ w \mapsto w+1 & & \end{array}$
 have that $A_0/f = P/f \cong \mathbb{C}/\mathbb{Z}$.
 But the left is l.h.p., right Euclidean.

Cor. The total number of attractors and periodic cycles is at most $2d-2$.

Different α 's, no finite set of invariant, Find projects

Functionally Indifferent Fixed Points

Julia / Fatou, Linearizability

Prop. $f(z_0) = z_0$, $\lambda = e^{2\pi i t}$, $t \notin \mathbb{Q}$. Then,
 $z_0 \in F(f)$ iff $\exists$ open $U \ni z_0$ and a lift $\tilde{f}: U \xrightarrow{\sim} \mathbb{D}$

$$\begin{array}{ccc}
 (*) & \begin{array}{ccc} \downarrow \tilde{f} & & \downarrow \tilde{f} \\ \mathbb{D} & \xrightarrow{\quad} & \mathbb{D} \\ w & \mapsto & \lambda w \end{array}
 \end{array}$$

Moreover, when this is the case, $(*)$ holds for the entire Fatou component containing z_0 .

* Such component are called Siegel disks.

if $z_0 \in J(f)$, then z_0 is a Cremer point.

Proof. $\Leftarrow$: $\{(w \mapsto \lambda w)\}_0^\infty$ is a normal family, so $\{f^n|_U\}$ is normal, hence $z_0 \in F(f)$.

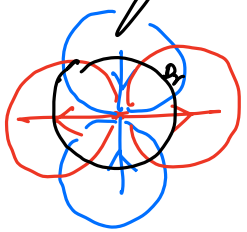
$\Rightarrow$: $z_0 \in F(f)$ is a Fatou component. Lift $\tilde{f}$:

$D \xrightarrow{\tilde{F}} D$ Let $\pi(\tilde{z}_0) = z_0$. Since $f(z_0) = z_0$, we suppose
 $\pi \downarrow \quad \downarrow \pi \quad \tilde{F}(\tilde{z}_0) = \tilde{z}_0$. ~~$D\pi|_{\tilde{z}_0} \cdot D\tilde{F}|_{\tilde{z}_0} = \lambda \cdot D\pi|_{\tilde{z}_0}$~~ By
 $u \xrightarrow{f} u = D/\Gamma$ following, $\tilde{F}$ is an inv. extn of D . Being
 Γ -equiv. means that $\tilde{F} \cdot \gamma = \gamma' \cdot \tilde{F}$. Can show that no
 such group Γ exists for $\tilde{F}$ inv. ext., so $\pi = \text{id}$.

No criterion, Existence? Only for an isolated pt.

Thm. Let $g_\lambda(z) = \lambda z + z^2$. For all λ in a suitable
 intersection of open dense set in the unit circle,
 there are inf. many periodic cycles in every
 neighborhood of 0 and hence 0 is a Cremona pt for g_λ .

Proof. $\lambda_0 = e^{2\pi i P/Q}$ (Small cycles property)



$\exists D_r$ s.t. g_{λ_0} has no cycles of period
 $< q$ in $D_r \setminus \{0\}$. This is an open condition.

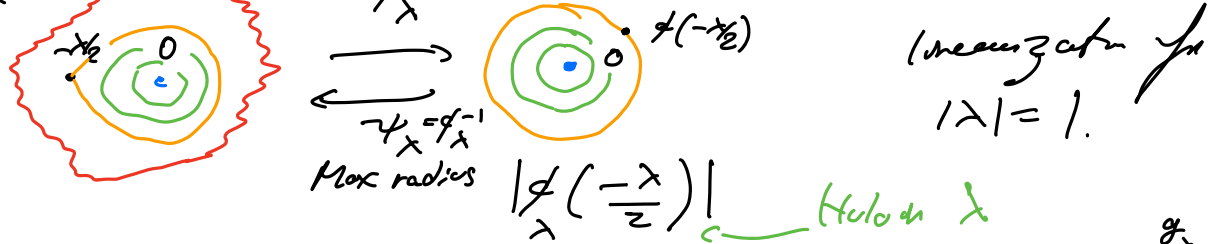
For λ near λ_0 , $g_\lambda^q(z) = z$ has q distinct solutions in D_r
 $\Rightarrow \exists U \ni \lambda_0$ open s.t. for $\lambda \in U \setminus \{\lambda_0\}$, g_λ has at
 least q -cycles in D_r . Take the intersection over all q of
 the union over all p of such $U \setminus \{\lambda_0\}$.

Thm. For λ in a set of full Lebesgue measure on
 the unit circle, g_λ has a Siegel disk at 0.

Proof. $|\lambda| = 1$:



$w \mapsto \lambda w$ - Want to find



$$\lambda \in \mathbb{C}$$

$$\sigma(\lambda) = \sup \left\{ r \geq 0 \mid \exists \text{ holo injective } \gamma: D_r \rightarrow \mathbb{C} \text{ s.t. } \gamma \uparrow \xrightarrow{g_\lambda} \mathbb{C} \text{ and } \gamma'(0) = 1 \right\}$$

$$\text{For } 0 < |\lambda| < 1, \sigma(\lambda) = |\phi_\lambda(-\frac{\lambda}{z})|. \sigma(0) = 0, \sigma(e^{2\pi i p/q}) = 0.$$

Note that if $\sigma(\lambda_0) > 0$ for $|\lambda_0| = 1$, then g_{λ_0} has a Siegel disk.

Claim: $\sigma: \mathbb{D} \rightarrow \mathbb{R}$ is bounded + upper semicont

$$1) \sigma(\lambda) \leq 3 \text{ because } \gamma \text{ must } \left\{ \begin{array}{l} \lim_{x \rightarrow x_0} \sigma(x) \leq \sigma(x_0) \\ \Leftrightarrow \\ \{x \mid \sigma(x) \geq r\} \text{ is closed} \end{array} \right.$$

map into $K(g_\lambda)$, but if $|z| > 3$,

$$|g_\lambda(z)| = |z^2 + \lambda z| \geq |z|^2 - |\lambda| \cdot |z| > 3|z| - |z| = 2|z|, g_\lambda(z) \rightarrow \infty.$$

$$2) \text{ If } \sigma(\lambda) \geq r, \text{ then } \exists \gamma_\lambda: D_r \rightarrow D_3 \text{ s.t. } (*)$$

But the set of such γ_λ is a normal family, so any limit pt. λ_0 has $\sigma(\lambda_0) \geq r$.

Thm. (F. and M. Riesz) If $\gamma: \mathbb{D} \rightarrow \mathbb{C}$ is nonconst, holo, and bounded, then for every $c_0 \in \mathbb{C}$,

$$\left\{ \theta \in \mathbb{R}/\mathbb{Z} \mid \lim_{r \rightarrow 1^-} \gamma(r \cdot e^{2\pi i \theta}) = c_0 \right\}$$

has Lebesgue measure zero.

$\Rightarrow$ For almost any θ ,

$$0 < \left| \lim_{r \rightarrow 1^-} f_{re^{2\pi i \theta}} \left(-\frac{re^{2\pi i \theta}}{z} \right) \right| = \lim_{r \rightarrow 1^-} \sigma(re^{2\pi i \theta}) \leq \sigma(e^{2\pi i \theta}).$$

Def. $r \in \mathbb{R}_{>0}$, $\exists ! a_0, a_1, \dots \in \mathbb{N}_{>0}$ s.t.

$$r = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}. \text{ Write } r = [a_0, a_1, a_2, \dots].$$

$$\frac{p_n}{q_n} = [a_0, \dots, a_n].$$

* Sequence is finite iff $r \in \mathbb{Q}$.

Thm. (Bergman) $\lambda = e^{2\pi i t}$, $t = [a_0, a_1, \dots]$. If

$$\sum_{n=1}^{\infty} \frac{\log \frac{q_{n+1}}{q_n}}{\frac{q_n}{q_1}} < \infty, \text{ then every fixed point of}$$

any rational mapping multiplies λ by a

Siegel disk.

$$q_n \sim q^n \frac{n \log q}{q^n}$$

$$\text{Ex: } f = \frac{1+\sqrt{5}}{2} = [1, 1, 1, \dots], \frac{p_n}{q_n} = \frac{F_{n+1}}{F_n}, \sum \frac{\log \frac{q_{n+1}}{q_n}}{\frac{q_n}{q_1}} \sim \sum \frac{n \log f}{f^n}$$

Share picture in Facebook

Thm. (Yoccoz) If $\sum_{n=1}^{\infty} \frac{\log \frac{q_{n+1}}{q_n}}{\frac{q_{n+1}}{q_1}}$ diverges, then

$g_\lambda(z) = \lambda z + z^2$ has a Cremona pt w/ the small cycles property.

Q: Is Yoccoz's thm true for any rational map and every fixed pt of mult λ ?

Q: Are there Cremer pts without the small cycles property?